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AN ALGORITHM TO COMPUTE PRIMARY DECOMPOSITION OF MONOMIAL IDEALS EQUIGENERATED IN DEGREE 2

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ABSTRACT. We give an algorithm to compute primary decomposition of monomial ideals equigenerated in degree 2 and establish connections with minimal vertex covers of a simple graph. We also describe an implementation in C++ of the algorithm.

1. Introduction

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field K. Let I be a monomial ideal whose minimal set of generators consists of monomials of degree 2.

When each element in the minimal set of generators is squarefree we have a natural link between I and a simple graph G on the vertex set $V(G) = \{x_1, \ldots, x_n\}$. The *edge ideal* $I := I(G)$, associated to G, is the ideal of S generated by the set of all squarefree monomials $x_i x_j$ so that x_i is adjacent to x_j in the graph G. For this ideal we know that exists a bijection between minimal prime ideals of $I(G)$ and minimal covers of G (see Ref. [1], Ch. 6).

We may consider the graph G' with loops such that $G \subset G'$, where the "loops" are the generators of degree 2 not squarefree. By this observation we give a natural generalization of the results given in [1] (see Theorem **3.12**).

In Section **2** we recall some necessary notions on graphs [2], graphs in commutative algebra [1] and simplicial complexes [3].

In Section **3** we recall the definition of irreducible irredundant primary decomposition of a monomial ideal and the results given in Chapter 6 of Ref. [1]. At the end of this section we prove the main Theorem of the article (see Theorem **3.12**) that strictly connects primary decomposition of $I(G')$ with prime decomposition of $I(G)$.

In Section **4** we recall the definition of Alexander duality [4] and give a way to compute the primary decomposition of the reduced ideal $I(G)$ by it (see Corollary **4.6**).

In Section **5** we provide efficient algorithms (see Algorithms 1, 3) to compute the primary decomposition of $I(G)$ and $I(G')$.

In the last section we describe an implementation in C++ of the algorithms 1, 3 written by the author under GPL license and freely downloadable [5].

2. Preliminaries

In this section we recall some definitions and properties on graphs and simplicial complexes that we shall use in the article.

Let G be a graph on the vertex set $V(G) = \{x_1 \ldots, x_n\}$ allowing loops and having no multiple edge.

Let $E(G) = \{\{x_i, x_j\} : x_i \text{ is adjacent to } x_j\}$ denote the set of edges of G and let $L(G) = \{x_l : x_l \text{ is a loop}\}$ denote the set of loops. If $L(G) = \emptyset$ we call the graph G *simple*.

A subset $C \subset V(G)$ is called a *vertex cover* of G if every edge and loop of G is incident with at least one vertex in C. A vertex cover C of G is called *minimal* if there is no proper subset of C which is a vertex cover of G. A subset A of $V(G)$ is called an *independent set* of G if no two vertices are adjacent. An independent set A of G is *maximal* if there exists no independent set which properly includes A. Notice that C is a minimal vertex cover of G if and only if $V(G) \setminus C$ is a maximal independent set of G.

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field K. The *edge ideal* $I(G)$, associated to G, is the ideal of S generated by the set of all squarefree monomials $x_i x_j$ so that x_i is adjacent to x_j.

Moreover $\text{height}\, I(G)$ is equal to the smallest number $|C|$ of vertices among all the minimal vertex covers C of G.

A graph G is called *unmixed* if all the minimal vertex covers of G have the same number of elements. A graph G is called *Cohen-Macaulay* if $S/I(G)$ is a Cohen-Macaulay ring.

We refer to Refs. [2] and [1] for detailed information on this subject.

Set $V = \{x_1, \ldots, x_n\}$. A *simplicial complex* Δ on the vertex set V is a collection of subsets of V such that (i)$\{x_i\} \in \Delta$ for all $x_i \in V$ and (ii)$F \in \Delta$ and $G \subseteq F$ imply $G \in \Delta$. An element $F \in \Delta$ is called a *face* of Δ. For $F \subset V$ we define the *dimension* of F by $\dim F = |F| - 1$, where $|F|$ is the cardinality of the set F. A maximal face of Δ with respect to inclusion is called a *facet* of Δ. If all facets of Δ have the same dimension, then Δ is called *pure*.

The Stanley-Reisner ideal of the simplicial complex Δ is the squarefree monomial

$$I_\Delta = (x_{i_1} \cdots x_{i_r} : 1 \leq i_1 < \ldots < i_r \leq n, \{x_{i_1}, \ldots, x_{i_r}\} \notin \Delta)$$

For an arbitrary graph G, we define the *complementary simplicial complex* of G by

$$\Delta(G) = \{A \subseteq V(G) : A \text{ is an independent set in} \, G\}.$$

Then $I(G)$ can be regarded as the Stanley-Reisner ideal $I_{\Delta(G)}$.

3. Primary decomposition

In this section we investigate primary decomposition of monomial ideals equigenerated in degree 2, namely, whose monomials in the minimal set of generators are all of degree 2.

An ideal Q is called *primary* if $xy \in Q$ implies either $x \in Q$ or $y^l \in Q$, $l \in \mathbb{N}$.

We recall that a primary decomposition of an ideal I is an expression of I as a finite intersection of primary ideals Q_i

$$I = \bigcap_{i=1}^{r} Q_i.$$

For arbitrary ideals is quite difficult to give an algorithm for the computation of the primary decomposition. The situation is different in case of monomial ideals. We give the following

Example 3.1. *Let $I = (x^2, xy)$ then*

$$I = (x) \cap (x^2, y) = (x) \cap (x^2, xy, y^k)$$

with $k \in \mathbb{N}$.

Remark 3.2. *By example **3.1** we observe that for a given ideal we may have infinitely many primary decompositions. Moreover the primary component*

$$(x^2, xy, y^k)$$

can be expressed

$$(x^2, xy, y^k) = (x^2, y) \cap (x, y^k).$$

Definition 3.3. *Let $S = K[x_1, \ldots, x_n]$ and let Q be a primary monomial ideal. Q is "irreducible" if it is generated by powers of variables that is*

$$Q = (x_{i_1}^{a_1}, x_{i_2}^{a_2}, \ldots, x_{i_r}^{a_r}),$$

with $1 \leq i_1 < i_2 < \cdots < i_r \leq n$, $a_i \in \mathbb{N}$, $i = 1, \ldots, r$.

Definition 3.4. *A primary decomposition that is an intersection of "irreducible" ideals is called "irreducible".*

Theorem 3.5. *Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K. If I is a monomial ideal then there exists a unique "irreducible" primary decomposition*

$$I = \bigcap_{i=1}^{r} Q_i.$$

Proof. See Corollary 5.1.13 and Theorem 5.1.17 of [1]. □

For the sake of completeness, we recall some known facts (see Ch. 6 of Ref. [1]).

Remark 3.6. *Let I be a monomial ideal equigenerated in degree 2 by squarefree monomial ideals. Then there exists a natural bijection between the set of generator of I*

$$\{x_{i_1}x_{j_1}, \ldots, x_{i_q}x_{j_q}\}$$

and the edges of a simple graph G

$$\{\{x_{i_1}, x_{j_1}\}, \ldots, \{x_{i_q}, x_{j_q}\}\}$$

where $\{x_{i_1}, x_{j_1}, \ldots, x_{i_q}, x_{j_q}\} \subseteq \{x_1, \ldots, x_n\}$.

Remark 3.7. *If $I(G)$ is squarefree it implies that each irredundant primary decomposition is an irreducible prime decomposition*

$$I(G) = \bigcap_{i=1}^{r} P_i$$

where each P_i is a prime ideal and is generated by a subset of $\{x_1, \ldots, x_n\}$.

The following nice fact holds (see Proposition 6.1.16, [1])

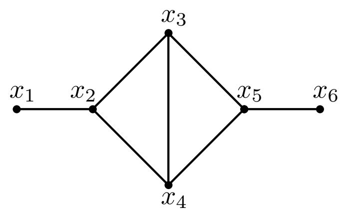
Theorem 3.8. *Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K, $I(G)$ an edge ideal. The following are equivalent:*

(1) *$P = (x_{i_1}, \ldots, x_{i_r})$ belongs to the irredundant prime decomposition of $I(G)$,*
(2) *$C = \{x_{i_1}, \ldots, x_{i_r}\}$ is a minimal vertex cover for G,*

where $\{x_{i_1}, \ldots, x_{i_r}\} \subset \{x_1, \ldots, x_n\}$.

Example 3.9. *Let $I = (x_1x_2, x_2x_3, x_2x_4, x_3x_4, x_3x_5, x_4x_5, x_5x_6)$ then*

$$I = (x_1, x_3, x_4, x_5) \cap (x_1, x_3, x_4, x_6) \cap (x_2, x_3, x_4, x_6) \cap (x_2, x_3, x_5) \cap (x_2, x_4, x_5).$$



The minimal vertex covers of the graph in the figure are in fact

$$\{x_1, x_3, x_4, x_5\}, \{x_1, x_3, x_4, x_6\}, \{x_2, x_3, x_4, x_6\}, \{x_2, x_3, x_5\}, \{x_2, x_4, x_5\}.$$

Our aim is to extend this result for a monomial ideal I equigenerated in degree 2 (not necessarily squarefree).

Remark 3.10. *Let*

$$I = \bigcap_{i=1}^{r} Q_i.$$

be an irreducible primary decomposition of the monomial ideal I equigenerated in degree 2. By Theorem **3.5** *and since $Q_i \supset I$ for all $i = 1, \ldots, r$, then*

$$Q_i = \sum_{j \in J_i} (x_j) + \sum_{l \in L_i} (x_l^2)$$

with $J_i \subset \{1, \ldots, n\}$, $|J_i| \geq 0$, $L_i \subset \{1, \ldots, n\}$, $|L_i| \geq 0$ and $J_i \cap L_i = \emptyset$.

To have a combinatorial interpretation of the monomials $x_i^2 \in I$ we consider a graph G' with loops and extend the concept of edge ideals to graphs with loops (see Ref. [6] and Example **3.13**).

Definition 3.11. *The edge ideal $I(G')$, associated to G', is the ideal of S generated by the set of all monomials x_ix_j so that $\{x_i, x_j\} \in E(G')$ and x_k^2 where $x_k \in L(G')$.*

Let G be the subgraph of G' without loops and isolated vertices. It is clear that $I(G) \subset I(G')$ (see Examples **3.9**, **3.13**).

Theorem 3.12. *Let I be a monomial ideal equigenerated in degree 2 and G' the graph with loops such that $I = I(G')$, G the subgraph of G' without loops and isolated vertices and let $I(G) = \bigcap_{i=1}^{r} P_i$ be its irredundant prime decomposition. Then the irreducible primary decomposition of $I = I(G')$ is $I = \bigcap_{i=1}^{r} Q_i$ with*

$$Q_i = P_i + \sum_{l \in L_i} (x_l^2)$$

such that $L_i = \{l : x_l \notin P_i, x_l \text{ is a loop of } G\}$.

Proof. By Definition **3.3** and Theorem **3.5** we have that Q_i, $i = 1, \ldots, r$ is primary and irreducible.

We want to prove that $Q_i \supset I$, $i = 1, \ldots, r$. It is sufficient to prove that for all minimal generators u of I we have that u is in Q_i for all i.

Since u is in the set of minimal generators of I and I is equigenerated in degree 2 either $u = x_j x_k$ with $j \neq k$ or $u = x_j^2$.

Let $u = x_j x_k$. In this case we have that u belongs to $I(G) \subset I(G')$ that is

$$u \in I(G) = \bigcap_{i=1}^{r} P_i$$

that is $u \in P_i \subset Q_i$ for all $i = 1, \ldots, r$.

Let $u = x_j^2$. If $x_j \in P_i$ we are done. Let $x_j \notin P_i$ by hypothesis $j \in L_i$ therefore $x_j^2 \in \sum_{l \in L_i} (x_l^2) \subset Q_i$.

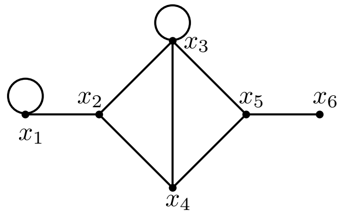
To finish the proof we need to show that $I = \bigcap_{i=1}^{r} Q_i$ is irredundant, that is we have to prove that for all $j \neq k$, $Q_j \not\subset Q_k$. If $Q_j \subset Q_k$ we also have $P_j \subset P_k$ contradicting the irredundancy of $I(G) = \bigcap_{i=1}^{r} P_i$. $\square$

Example 3.13. *Let*

$$I = (x_1^2, x_1x_2, x_2x_3, x_2x_4, x_3^2, x_3x_4, x_3x_5, x_4x_5, x_5x_6)$$

then

$$\begin{aligned} I \; = \; & (x_1, x_3, x_4, x_5) \cap (x_1, x_3, x_4, x_6) \cap \\ & \cap (x_1^2, x_2, x_3, x_4, x_6) \cap (x_1^2, x_2, x_3, x_5) \cap (x_1^2, x_2, x_3^2, x_4, x_5). \end{aligned}$$



4. Alexander duality

The primary decomposition of monomial ideals equigenerated in degree 2 by Theorem **3.12** is induced by the primary decomposition of squarefree monomial ideals. In this section we give a way to compute it by Alexander duality.

Let G be a simple graph and $\Delta(G)$ be the simplicial complex of the Stanley-Reisner ideal $I(G)$ (see Section **2**).

We recall the following (see Ref. [4], Definition 1.35):

Definition 4.1. *Let* $I = (\mathbf{x}^{\alpha_1}, \dots, \mathbf{x}^{\alpha_q}) \subset K[\mathbf{x}] = K[x_1, \dots, x_n]$ *be a square-free monomial ideal, with* $\alpha_i = (\alpha_{i_1}, \dots, \alpha_{i_n}) \in \{0,1\}^n$*. The* Alexander dual *of* I *is the ideal*

$$I^* = \bigcap_{i=1}^{q} \mathfrak{m}_{\alpha_i},$$

where $\mathfrak{m}_{\alpha_i} = (x_j : \alpha_{i_j} = 1)$.

We recall the following (see Ref. [4], Proposition 1.37):

Proposition 4.2. *If* Δ *is a simplicial complex, then its Alexander dual is* Δ^**, consisting of the complements of the non faces of* Δ.

The Alexander duality really is a duality (see Ref. [4], Proposition 5.1):

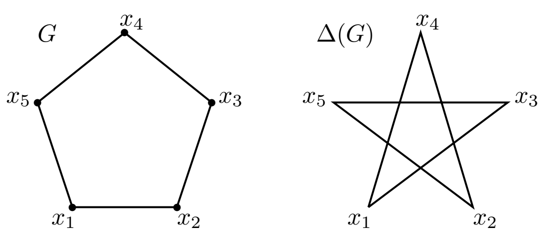
Proposition 4.3. *If* I *is a squarefree monomial ideal then* $(I^*)^* = I$*. Equivalently* $(\Delta^*)^* = \Delta$*. Moreover* $I_{\Delta^*} = I^*_{\Delta}$.

By Definition **4.1** and Proposition **4.3** we obtain the following

Remark 4.4. *If* I *is a squarefree monomial ideal then*

$$I = (\bigcap_{i=1}^{q} \mathfrak{m}_{\alpha_i})^*$$

Example 4.5. *Let* $I(G) = (x_1x_2, x_1x_5, x_2x_3, x_3x_4, x_4x_5)$*. It is easy to observe that the facets of* $\Delta(G)$ *are* $\{x_1, x_3\}$, $\{x_1, x_4\}$, $\{x_2, x_4\}$, $\{x_2, x_5\}$, $\{x_3, x_5\}$ *(see figure)*



and

$$\begin{aligned} I(G)^* &= (x_1,x_2) \cap (x_1,x_5) \cap (x_2,x_3) \cap (x_3,x_4) \cap (x_4,x_5) \\ &= (x_1x_2x_4, x_1x_3x_4, x_1x_3x_5, x_2x_3x_5, x_2x_4x_5) \end{aligned}$$

whose generators represent the minimal covers of G*. The prime decomposition of* $I(G)$ *is*

$$I(G) = (x_1,x_2,x_4) \cap (x_1,x_3,x_4) \cap (x_1,x_3,x_5) \cap (x_2,x_3,x_5) \cap (x_2,x_4,x_5).$$

To complete the picture we give also the facets of Δ^* *:*

$$\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_3, x_4, x_5\}, \{x_1, x_4, x_5\}, \{x_1, x_2, x_5\}.$$

Therefore we have a nice algebraic method to compute the primary decomposition of $I(G)$ that is

Corollary 4.6. *Let* I *be a squarefree monomial ideal equigenerated in degree* 2 *on the polynomial ring* $S = K[x_1, \dots, x_n]$*,* $I = (x_{i_1}x_{j_1}, \dots, x_{i_q}x_{j_q})$*, where*

$$\{x_{i_1}, x_{j_1}, \dots, x_{i_q}, x_{j_q}\} \subseteq \{x_1, \dots, x_n\}.$$

Then its irredundant irreducible primary decomposition is represented by the minimal set of generators of I^**:*

$$I^* = (x_{i_1}, x_{j_1}) \cap (x_{i_2}, x_{j_2}) \cap \dots \cap (x_{i_q}, x_{j_q}).$$

5. The algorithm

The calculation of primary decomposition of monomial ideals equigenerated in degree 2 is strictly related with the research of all minimal vertex covers of a graph G, but neither approximation method nor partial answer are enough. In this section we give an algorithm and we make some remarks on the complexity.

We recall that following

Definition 5.1. *NP is the set of decision problems solvable in polynomial time by a non-deterministic Turing machine.*

Definition 5.2. *A decision problem* C *is NP-complete if:*

(1) C *is in NP,*
(2) *Every problem in NP is reducible to* C *in polynomial time.*

Proposition 5.3. *The following problem is NP-complete:*

"Let I *be a monomial ideal equigenerated in degree* 2 *over the polynomial ring* $S = K[x_1, \dots, x_n]$ *and let* $h \in \mathbb{N}$*,* $h < n$*. Is* height $I \leq h$*?"*

Proof. To prove that a problem is NP-complete is sufficient to see if some of the known NP-complete problem can be reduced to it. By Theorem **3.12**, the fact that the height of I is less than or equal to h is equivalent to the existence of a vertex cover C of the graph G related to I such that its cardinality is less than or equal to h. This is a famous NP-complete problem. □

Remark 5.4. *By using Corollary* ***4.6*** *the computation of such intersection of ideals needs* $O(2^q)$ *operations (the least common multiple of the generators) and the most of the computed generators will be not minimal.*

For example if $I = I(K_n)$ *where* K_n *is a complete graph with* n *vertices we will have* $2^{\frac{n(n-1)}{2}}$ *generators and the minimal covers are exactly* n*, each of them with* $n-1$ *elements.*

A more efficient method is the following. We calculate in each step k*,* $k = 1, \dots, q$*, the minimal set of generators of he subideal* I_k^* *where:*

$$I_k^* = (x_{i_1}, x_{j_1}) \cap (x_{i_2}, x_{j_2}) \cap \dots \cap (x_{i_k}, x_{j_k}).$$

by the observation $I_k^* = I_{k-1}^* \cap (x_{i_k}, x_{j_k})$. In this way we control the growth of not minimal generators.

By a different point of view the minimal set of generators of I_k^* represents the minimal covers of the graph G_k, that is a subgraph of G on the first k edges.

By Corollary **4.6** and Remark **5.4** we have Algorithm 1, the function Minimize deletes all the not minimal generators in M (see Algorithm 2) and by Theorem **3.12** we obtain Algorithm 3.

Algorithm 1: Prime decomposition

Data: $I(G) = (x_{k1}x_{k2} : 1 \leq k1 < k2 \leq q_1, k1 < k2)$
Result: $M = \{$ minimal set of generators of $I^* \}$
$M = \{x_{11}, x_{12}\}$;
for $k = 2, \ldots, q_1$ **do**
  foreach $C \in M$ **do**
    if $x_{k1} \nmid C \wedge x_{k2} \nmid C$ **then**
      $C_1 := C \cdot x_{k2}$;
      $C := C \cdot x_{k1}$;
      Minimize(M,C);
      $M := M \cup \{C_1\}$;
      Minimize(M,C_1);
    else
      Minimize(M,C);

Algorithm 2: Minimize

Data: $M = \{C_1, \ldots, C_r\}, C$
Result: $M = \{$ minimal generators $\}$
$i := 1$;
while $C_i \neq C$ **do**
  if $C_i < C \wedge C_i \mid C$ **then**
    $M := M \setminus \{C\}$;
    return
  else if $C < C_i \wedge C \mid C_i$ **then**
    $M := M \setminus \{C_i\}$;
  $i := i + 1$;

Algorithm 3: Primary decomposition

Data: $I(G') = (x_{k1}x_{k2} : 1 \leq k1 < k2 \leq q_1, k1 < k2) + (x_l^2 : 1 \leq l < q_2)$
Result: $Q = \{$ irreducible components of $I\}$

Calculate P by Algorithm 1;
$Q = \emptyset$;
foreach $C \in P$ **do**
  $Q_C := C$;
  for $l = 1, \ldots, q_2$ **do**
    if $x_l \notin C$ **then**
      $Q_C := Q_C \cup \{x_{kj}\}$;
  $Q := Q \cup \{Q_C\}$;

6. C++ implementation

In this section we describe some interesting features and choices in the implementation in C++ [5] of the algorithms shown in Section **5**.

We choose the C++ language for three main reasons:

(1) By its object-oriented features we overload the standard operand (like $<$, $>$) and we can easily extend the software: for example the computation of primary decomposition of every monomial ideal;
(2) The bit-oriented environment increases the speed of computation;
(3) By the STL library provided in ANSI C++ it is easy to implement lists of objects (in our case monomials).

In particular we give only some examples related to the Class bit_int

```
class bit_int {
private:
 unsigned int bi;
 unsigned int sz;
public:
 static unsigned int nrbit;
 bit_int(){}
 bit_int(unsigned int Bi ){bi=Bi;sz=1;}
 ...
 ...
 void join(bit_int bi1){bi=bi|bi1.get();sz++;};
 void join(bit_int bi1,bit_int bi2)
               {bi=bi1.get()|bi2.get();sz++;};
 int size(){return sz;}
 bool operator<(bit_int &bi2){return sz<bi2.size();}
 bool operator>(bit_int &bi2){return sz>bi2.size();}
 bool operator==(bit_int &bi2){return sz==bi2.size();}
};
```

In line 03 we see that our class use an "unsigned int", that is we have an upper bound on the number of indeterminates n of our ring $K[x_1, \ldots, x_n]$ (in our implementation $n \leq 32$). The class bit_int makes it possible to represent a squarefree monomial by a sequence of bits.

Example 6.1. *If $n = 7$ we have the following translation*

$$x_1x_2x_4x_7 = 1101001$$

Even if we have this limitation on the other hand we earn a fast computation.

In line 04 we define the size (sz) of our covering or equivalently the degree of the monomial in I^* so that we have this value pre-computed in the class.

In line 11 we implement the multiplication of two monomials: by operators like $join$ we make a "parallel" computation of the product since we use the bit-oriented "or" provided by C++. The operator $join$ also increases by one the degree since the second monomial in the product is always of degree 1 (see Algorithm 1).

Example 6.2. *If $n = 7$ we have the following translation with $C_1 = x_1x_2x_4x_7$, $C_2 = x_3$*

$$\begin{array}{cc} 1101001 & + \\ 0010000 & = \\ \hline \\ 1111001 & \end{array}$$

A comparison table with the well known software like CoCoA and Macaulay2 are meaningless since they use an interpreter and provide algorithms to compute the primary decomposition over general monomial ideals. In any case computation of all the graphs with 10 vertices in these software systems can take days against our implementation that takes only 20 minutes.

In the following table we give the time of computation of primary decompositions of squarefree ideals induced by all graphs with n vertices.

n	Nr.Ideals	Time (sec.)
6	112	< 1
7	853	< 1
8	11.117	2
9	261.080	23
10	11.716.571	1.247

References

[1] R. H. Villarreal. *Monomial Algebras*, Pure and Applied Mathematics (Marcel Dekker, New York/Basel, 2001).

[2] R. Diestel, *Graph Theory*, 2nd edition, GTM **173** (Springer, Berlin/Heidelberg/New York, 2000).

[3] R. P. Stanley, *Combinatorics and commutative algebra*, 2nd edition, (Birkäuser, Boston/ Basel/ Stuttgart, 1996).

[4] E. Miller and B. Sturmfels, *Combinatorial Commutative Algebra* (Springer-Verlag, Berlin, 2005).

[5] G. Rinaldo, *PrimDec2: Software to compute primary decomposition of monomial ideals equigenerated in degree* 2; available at the URL: `http://ww2.unime.it/algebra/rinaldo/primdec2`.

[6] G. Molica Bisci and G. Rinaldo, *Cohen-Macaulay graphs with loops*, Seminarberichte aus dem Fachbereich Mathematik der FernUniversitaet Hagen, Vol. 76, 2004.

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